SOLUTION OF CERTAIN VARIATIONAL PROBLEMS OF THERMAL RESILIENCE FOR THIN SHELLS CONSIDERING THE SELECTION OF OPTIMUM CONDITIONS FOR LOCALIZED HEAT TREATMENT

É. I. Grigolyuk, Ya. I. Burak, and Ya. S. Podstrigach

One of the possible ways of stating and solving the selection problem for optimum temperature fields for localized axisymmetric heating of shells is investigated. The minimum of shell elastic energy is taken as the optimization criterion. An infinite cylindrical shell was considered in a similar formulation in [1]. The corresponding variational problem is formulated for the functional of elastic energy with additional limitations imposed on the function of twist angle at specified shell sections. The variational problem is reduced to an isoperimetric by the use of singular functionals of the δ-function kind. The related Euler equation is obtained, and this together with the problem resolvent equation constitute a complete set of equations for determining the extremum temperature field with related stress-strain state of the shell. Cylindrical, conical, and spherical shells are considered separately. A numerical analysis is made for the simplest conditions of localized heating of cylindrical and conical shells.

1. Let a shell of revolution, represented in canonical coordinates of its primary curvatures, be under the influence of an axisymmetric temperature field. In the absence of external forces, the problem of defining the stress-strain state of the shell for a given temperature field can then be reduced to finding the function of twist angles which would satisfy the resolvent equation

$$\left\{\left[\frac{d}{ds}\left(\frac{d}{ds}+\frac{r^{\cdot}}{r}\right)\frac{1}{k_2}+(1+\nu)k_1\right]\left[\frac{d}{ds}\left(\frac{d}{ds}+\frac{r^{\cdot}}{r}\right)+(1-\nu)k_1k_2\right]+mk_2\right\}\theta = \\ =\alpha m\frac{dT}{ds},$$

$$m=D_0/D_1,\quad D_0=2Eh,\quad D_1={}^2/_3 Eh^3/(1-\nu^2). \tag{1.1}$$

Here θ (s) is a function of twist angle of the shell median surface; T(s) is the temperature; s is the length of the meridian arc measured from a given section; r=r(s) is a cross-section radius; k_1 and k_2 are the curvatures of meridians and parallels, respectively; D_0 is the rigidity in tension; D_1 the torsional rigidity; E is the modulus of elasticity; ν is the Poisson coefficient; 2h is the shell thickness; α is the coefficient of thermal expansion. The dot over a magnitude denotes a derivative with respect to arc s. For the determination of θ Eq. (1.1) must be supplemented by suitable conditions at the ends of the shell. When function θ(s) has been found, nonzero stresses N_1 and N_2, moments M_1 and M_2, and the strain components ε_1, ε_2, $\varkappa_1$, and $\varkappa_2$ of the median surface of the shell are defined by

---

Moscow, L'vov. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 9, No. 4, pp. 47-54, July-August, 1968. Original article submitted February 7, 1968.

$$N_1 = -D_1 \frac{r^{\cdot}}{k_2 r}\left[\frac{d}{ds}\left(\frac{d}{ds} + \frac{r^{\cdot}}{r}\right) + (1-\nu) k_1 k_2\right]\theta,$$
$$N_2 = -D_1 \frac{d}{ds}\left\{\frac{1}{k_2}\left[\frac{d}{ds}\left(\frac{r^{\cdot}}{r} + \frac{d}{ds}\right) + (1-\nu) k_1 k_2\right]\right\}\theta,$$
$$M_1 = -D_1\left(\frac{d}{ds} + \nu\frac{r^{\cdot}}{r}\right)\theta, \quad M_2 = -D_1\left(\nu\frac{d}{ds} + \frac{r^{\cdot}}{r}\right)\theta,$$
$$\varepsilon_1 = \frac{1}{m}\left(\nu\frac{d}{ds} - \frac{r^{\cdot}}{r}\right)\left[\frac{1}{k_2}\frac{d}{ds}\left(\frac{d}{ds} + \frac{r^{\cdot}}{r}\right) + (1-\nu) k_1\right]\theta + \alpha T,$$
$$\varepsilon_2 = \frac{1}{m}\left(\nu\frac{r^{\cdot}}{r} - \frac{d}{ds}\right)\left[\frac{1}{k_2}\frac{d}{ds}\left(\frac{d}{ds} + \frac{r^{\cdot}}{r}\right) + (1-\nu) k_1\right]\theta + \alpha T,$$
$$\varkappa_1 = -\theta^{\cdot}, \qquad \varkappa_2 = -\frac{r^{\cdot}}{r}\theta. \tag{1.2}$$

Here subscripts 1 and 2 denote magnitudes in the meridional and the parallel directions, respectively. We introduce into our considerations the elastic energy of the shell [1]

$$K = \iint_{(S)} (N_1\varepsilon_1{}^0 + N_2\varepsilon_2{}^0 + M_1\varkappa_1{}^0 + M_2\varkappa_2{}^0)\, dS, \tag{1.3}$$

$$\varepsilon_1{}^0 = \varepsilon_1 - \alpha T, \quad \varepsilon_2{}^0 = \varepsilon_2 - \alpha T, \quad \varkappa_1{}^0 = \varkappa_1, \quad \varkappa_2{}^0 = \varkappa_2. \tag{1.4}$$

Here S is the median surface of the shell, and ε_1^0, ε_2^0, $\varkappa_1^0$, and $\varkappa_2^0$ are elastic stain components.

The integrand in (1.3) is a positively defined quadratic form with respect to stresses and moments, if and only if the temperature-induced stresses are zero. It is therefore natural, when solving a problem of localized heating which results in a comparatively low level of temperature-induced stresses, to take the minimum of functional (1.3) of the shell elastic energy as the integral condition of optimization.

Substituting (1.2) and (1.4) into (1.3), we obtain

$$K(\theta) = \frac{4\pi D_1}{m}\int_{(L)} F(s, \theta, \theta^{\cdot}, \theta^{\cdot\cdot}, \theta^{\cdot\cdot\cdot})\, ds. \tag{1.5}$$

Here (L) is the meridional line, and the following notation has been used

$$F = \frac{r}{2}\left\{V^{\cdot 2} - 2\nu\frac{r^{\cdot}}{r}V^{\cdot}V + \left(\frac{r^{\cdot}}{r}\right)^2 V^2 + m\left[\theta^{\cdot 2} + 2\nu\frac{r^{\cdot}}{r}\theta^{\cdot}\theta + \left(\frac{r^{\cdot}}{r}\right)^2\theta^2\right]\right\},$$
$$V = \frac{1}{k_2}\left[\frac{d}{ds}\left(\frac{d}{ds} + \frac{r^{\cdot}}{r}\right) + (1-\nu) k_1 k_2\right]\theta. \tag{1.6}$$

The variational problem is formulated as follows. First, we have to find the extremum of functional $K[\theta]$ for the set of functions $\theta=\theta(s)$ satisfying the following conditions:

1) at specified sections $s=s_j$ $(j=1, 2, \ldots, n)$

$$\frac{d^{(i)}\theta(s_j)}{ds^i} = \theta_{ij}, \qquad \int_{s_0}^{s_j}\theta(s)\, ds = \theta_j \qquad (i=0, 1, 2); \tag{1.7}$$

2) at end sections $s=s_0$ and $s=s_*$

$$\frac{d^{(i)}\theta(s_0)}{ds^i} = \theta_{i0}, \qquad \frac{d^{(i)}\theta(s_*)}{ds^i} = \theta_{i*} \quad (i=0, 1, 2). \tag{1.8}$$

Here θ_{ij} and θ_j are arbitrary numbers which can be further defined by specifying at sections $s=s_j$ numerical values for the problem parameters (temperature, stresses, moments, etc.). It should be noted that θ_{i0} and θ_{i*} must be bound by additional relationships owing to conditions at the free ends. The stated problem is equivalent to this isoperimetric problem.

Second, we have to find the extremum of functional $K[\theta]$ for the set of functions $\theta=\theta(s)$ whose functionals

$$K_{ij}[\theta] = (-1)^i \int_{s_0}^{s_*} \delta^{(i)}(s-s_j)\,\theta(s)\,ds, \qquad K_j[\theta] = \int_{s_0}^{s_*} S_+(s_j-s)\,\theta(s)\,ds, \tag{1.9}$$

in which $\delta^{(i)}(s)$ is the i-th derivative of the delta function and $S_+(s)$ is the jump function, assuming the specified values

$$K_{ij}[\theta] = \theta_{ij}, \qquad K_j[\theta] = \theta_j. \tag{1.10}$$

It is assumed here that each function of the considered set satisfies (1.8)

Such a problem reduces to finding the extremum of functional [3]

$$K^*[\theta] = \frac{4\pi D_1}{m}\int_{s_0}^{s_*}\Big\{F(s,\theta,\theta^{\cdot},\theta^{\cdot\cdot},\theta^{\cdot\cdot\cdot}) - \theta(s)\Big[\sum_{j=1}^{n}\Big(\sum_{i=0}^{2}\lambda_{ij}\delta^{(i)}(s-s_j) + \lambda_j S_+(s_j-s)\Big)\Big]\Big\}ds. \tag{1.11}$$

Here λ_{ij} and λ_j are arbitrary constants which ensure the fulfillment of (1.7). The Euler equation for the functional $K^*(\theta)$ yields

$$\frac{\partial F}{\partial\theta} - \frac{d}{ds}\left(\frac{\partial F}{\partial\theta^{\cdot}}\right) + \frac{d^2}{ds^2}\left(\frac{\partial F}{\partial\theta^{\cdot\cdot}}\right) - \frac{d^3}{ds^3}\left(\frac{\partial F}{\partial\theta^{\cdot\cdot\cdot}}\right) = \sum_{j=1}^{n}\Big[\sum_{i=0}^{2}\lambda_{ij}\delta^{(i)}(s-s_j) + \lambda_j S_+(s_j-s)\Big], \tag{1.12}$$

where

$$\begin{aligned}
\frac{\partial F}{\partial\theta} &= \Big[V^{\cdot}\Big(\nu r^{\cdot} - r\frac{d}{ds}\Big) + \frac{r^{\cdot}}{r}V\Big(\nu r\frac{d}{ds} - r^{\cdot}\Big)\Big]\Big(\frac{r^{\cdot 2}}{k_2 r^2} + \nu k_1\Big) + \frac{m r^{\cdot}}{r}(r^{\cdot}\theta + \nu r\theta^{\cdot})\\
\frac{\partial F}{\partial\theta^{\cdot}} &= (\nu r^{\cdot}V - rV^{\cdot})\Big[(1+\nu)k_1 + \frac{r^{\cdot}}{k_2 r}\Big(\frac{k^{\cdot}_2}{k_2} + 2\frac{r^{\cdot}}{r}\Big)\Big] + \frac{r^{\cdot 2}}{k_2 r^2}(r^{\cdot}V - \nu rV^{\cdot}) + m(r\theta^{\cdot} + \nu r^{\cdot}\theta),\\
\frac{\partial F}{\partial\theta^{\cdot\cdot}} &= \frac{r^{\cdot}}{k_2}\Big[\Big(2 - \nu - \frac{k_1}{k_2}\Big)V^{\cdot} - \frac{r^{\cdot}}{r}\Big(2\nu - 1 - \nu\frac{k_1}{k_2}\Big)V\Big]\\
\frac{\partial F}{\partial\theta^{\cdot\cdot\cdot}} &= \frac{r}{k_2}\Big(V^{\cdot} - \nu\frac{r^{\cdot}}{r}V\Big),\\
V^{\cdot} = \frac{dV}{ds} &= \frac{1}{k_2}\Big\{\theta^{\cdot\cdot\cdot} + \Big(2 - \frac{k_1}{k_2}\Big)\frac{r^{\cdot}}{r}\theta^{\cdot\cdot} - \Big[(1+\nu)k_1k_2 + \frac{r^{\cdot}}{r}\Big(\frac{k^{\cdot}_2}{k_2} + 2\frac{r^{\cdot}}{r}\Big)\Big]\theta^{\cdot} - k_2\theta\frac{d}{ds}\Big(\frac{r^{\cdot 2}}{k_2 r^2} + \nu k_1\Big)\Big\}.
\end{aligned} \tag{1.13}$$

Equation (1.12) together with the resolvent equation (1.1) and (1.7) and (1.8) constitute the complete system of relationships defining the temperature field and the related stress-strain state of the shell.

2. Let us write the fundamental equations for cylindrical, conical, and spherical shells.

a) Cylindrical Shell. ($k_1=0$, $k_2=1/R$, $r=R$, and $\dot{r}=0$). Equations (1.1) and (1.12) with (1.2) written in terms of the axial x-coordinate are of the form

$$\frac{d^4\theta}{dx^4} + \frac{m}{R^2}\theta = \frac{\alpha m}{R}\frac{dT}{dx}, \tag{2.1}$$

$$\frac{d^6\theta}{dx^6} + \frac{m}{R^2}\frac{d^2\theta}{dx^2} = -\frac{1}{R^3}\sum_{j=1}^{n}\Big[\sum_{i=0}^{2}\lambda_{ij}\delta^{(i)}(x-x_i) + \lambda_j S_+(x_j-x)\Big], \tag{2.2}$$

$$N_1 = 0, \quad N_2 = -D_1 R \frac{d^3\theta}{dx^3}, \quad M_1 = -D_1 \frac{d\theta}{dx}, \quad M_2 = -\nu D_1 \frac{d\theta}{dx}. \tag{2.3}$$

By virtue of (2.1) we obtain from (2.2) the following equation

$$\frac{d^3T}{dx^3} = -\frac{1}{\alpha m R^2} \sum_{j=1}^{n} \left[\sum_{i=0}^{2} \lambda_{ij} \delta^{(i)}(x - x_j) + \lambda_j S_+(x_j - x)\right], \tag{2.4}$$

by which it is possible to determine directly extremum temperature fields.

b) Conical Shell. We denote by β the angle between the shell axis and the generatrix of its median surface. The s-coordinate will be measured from the cone vertex. Then

$$k_1 = 0, \quad k_2 = \frac{\operatorname{ctg}\beta}{s}, \quad r = s \sin\beta, \quad \frac{r'}{r} = \frac{1}{s}. \tag{2.5}$$

Substituting these values into Eqs. (1.1), (1.11), and (1.2), we obtain

$$s\frac{d^4\theta}{ds^4} + 4\frac{d^3\theta}{ds^3} + \frac{m \operatorname{ctg}^2\beta}{s}\theta = \alpha m \operatorname{ctg}\beta \frac{dT}{ds}, \tag{2.6}$$

$$\frac{d}{ds}\left\{\frac{1}{s}\frac{d}{ds}\left[s^2\left(s\frac{d^4\theta}{ds^4} + 4\frac{d^3\theta}{ds^3} + \frac{m \operatorname{ctg}^2\beta}{s}\theta\right)\right]\right\} = -\frac{\operatorname{ctg}^2\beta}{s \sin\beta} \sum_{j=1}^{n} \left[\sum_{i=0}^{2} \lambda_{ij}\delta^{(i)}(s - s_j) + \lambda_j S_+(s_j - s)\right], \tag{2.7}$$

$$N_1 = -D_1 \operatorname{tg}\beta\left(\frac{d^2\theta}{ds^2} + \frac{1}{s}\frac{d\theta}{ds} - \frac{1}{s^2}\theta\right),$$

$$N_2 = D_1 \operatorname{tg}\beta\left(s\frac{d^3\theta}{ds^3} + 2\frac{d^2\theta}{ds^2} - \frac{1}{s}\frac{d\theta}{ds} + \frac{1}{s^2}\theta\right). \tag{2.8}$$

Comparison of the left-hand sides of Eqs. (2.6) and (2.7) shows that extremum temperature fields of a conical shell are defined by an equation similar to (2.4)

$$\frac{d}{ds}\left[\frac{1}{s}\frac{d}{ds}\left(s^2\frac{dT}{ds}\right)\right] = -\frac{\operatorname{ctg}\beta}{\alpha m \sin\beta}\frac{1}{s}\sum_{j=1}^{n}\left[\sum_{i=0}^{2}\lambda_{ij}\delta^{(i)}(s - s_j) + \lambda_j S_+(s_j - s)\right]. \tag{2.9}$$

c) Spherical Shell. In this case

$$k_1 = k_2 = \frac{1}{R}, \quad s = R\varphi, \quad r = R\sin\varphi, \quad \frac{r'}{r} = \frac{1}{R}\operatorname{ctg}\varphi. \tag{2.10}$$

Here R is the radius of the shell curvature and φ is the meridian arc measured from the axis of revolution. The resolvent equation (1.1) and the Euler equation (1.12) can now be reduced to the form

$$\left[\left(\frac{d^2}{d\varphi^2} + \operatorname{ctg}\varphi\frac{d}{d\varphi} - \operatorname{ctg}^2\varphi + \nu\right)\left(\frac{d^2}{d\varphi^2} + \operatorname{ctg}\varphi\frac{d}{d\varphi} - \operatorname{ctg}^2\varphi - \nu\right) + mR^2\right]\theta = \alpha m R^2 \frac{dT}{d\varphi}, \tag{2.11}$$

$$\left(\frac{d^2}{d\varphi^2} + \operatorname{ctg}\varphi\frac{d}{d\varphi} - \operatorname{ctg}^2\varphi - \nu\right)\left[\left(\frac{d^2}{d\varphi^2} + \operatorname{ctg}\varphi\frac{d}{d\varphi} - \operatorname{ctg}^2\varphi + \nu\right)\left(\frac{d^2}{d\varphi^2} + \operatorname{ctg}\varphi\frac{d}{d\varphi} - \operatorname{ctg}^2\varphi - \nu\right) + mR^2\right]\theta = -\frac{1}{\sin\varphi}\sum_{j=1}^{n}\left[\sum_{i=0}^{2}\lambda_{ij}R^{2-i}\delta^{(i)}(\varphi - \varphi_j) + \lambda_j S_+(\varphi_j - \varphi)\right]. \tag{2.12}$$

The expressions defining stresses N_1 and N_2 and moments M_1 and M_2 are

$$N_1 = -\frac{D_1 \operatorname{ctg}\varphi}{R^2}\left[\frac{d^2\theta}{d\varphi^2} + \operatorname{ctg}\varphi\frac{d\theta}{d\varphi} - (\nu + \operatorname{ctg}^2\varphi)\,\theta\right],$$

$$N_2 = -\frac{D_1}{R^2}\left[\frac{d^3\theta}{d\varphi^3} + \operatorname{ctg}\varphi\frac{d^2\theta}{d\varphi^2} - (1 + \nu + 2\operatorname{ctg}^2\varphi)\frac{d\theta}{d\varphi} + \frac{2\operatorname{ctg}\varphi}{\sin^2\varphi}\,\theta\right],$$

$$M_1 = -\frac{D_1}{R}\left(\frac{d\theta}{d\varphi} + \nu\operatorname{ctg}\varphi\theta\right), \quad M_2 = -\frac{D_1}{R}\left(\operatorname{ctg}\varphi\theta + \nu\frac{d\theta}{d\varphi}\right). \tag{2.13}$$

To determine coefficients λ_{ij} and λ_j, and the constants of integration it is necessary in each of these cases to supplement the derived relationships by (1.7) and (1.8) or conditions corresponding to other parameters of the problem.

Solutions of this problem for an infinite cylindrical shell and for a conical shell closed at the vertex are given below.

3. To determine the extremum solution in an infinite cylindrical shell we use (2.1) and (2.4) as the input equations, and write these in the form

$$\frac{d^4\theta}{d\xi^4} + 4\theta = 4\alpha a\frac{dT}{d\xi} \qquad \left(\xi = a\frac{x}{R},\ a^4 = \frac{3(1-\nu^2)R^2}{4h^2}\right), \tag{3.1}$$

$$\frac{d^3T}{d\xi^3} = \sum_{j=1}^{n}\left[\sum_{i=0}^{2}\gamma_{ij}\delta^{(i)}(\xi - \xi_j) - \gamma_j S_+(\xi_j - \xi)\right]. \tag{3.2}$$

Here γ_{ij} and γ_j are arbitrary constants.

Let us find the solution of Eqs. (3.1) and (3.2) that vanishes at infinity. With the use of the Fourier transformation we obtain [4]

$$T = \frac{1}{2}\sum_{j=1}^{n}\left[\frac{\gamma_j}{6}(\xi - \xi_j)^3 + \frac{\gamma_{0j}}{2}(\xi - \xi_j)^2 + \gamma_{1j}(\xi - \xi_j) + \gamma_{2j}\right]\operatorname{sgn}(\xi - \xi_j), \tag{3.3}$$

$$\theta = \frac{\alpha a}{2}\sum_{j=1}^{n}\left\{\left[\frac{\gamma_j}{3}(\xi - \xi_j)^2 + \gamma_{0j}(\xi - \xi_j) + \gamma_{1j}\right]\operatorname{sgn}(\xi - \xi_j) + \right.$$
$$+ e^{-|\xi - \xi_j|}\left[\frac{\gamma_j}{2}\sin(\xi - \xi_j) + \frac{\gamma_{0j}}{2}(\cos(\xi - \xi_j) - \sin|\xi - \xi_j|) - \right.$$
$$\left.\left. - \gamma_{1j}\operatorname{sgn}(\xi - \xi_j)\cos(\xi - \xi_j) + \gamma_{2j}(\cos(\xi - \xi_j) + \sin|\xi - \xi_j|)\right]\right\}. \tag{3.4}$$

The coefficients γ_{ij} and γ_j must now satisfy the relationships

$$\sum_{j=1}^{n}\gamma_j = 0, \quad \sum_{j=1}^{n}(\gamma_j\xi_j - \gamma_{0j}) = 0, \quad \sum_{j=1}^{n}(\gamma_j\xi_j^2 - 2\gamma_{0j}\xi_j + \gamma_{1j}) = 0,$$
$$\sum_{j=1}^{n}(\gamma_j\xi_j^3 - 3\gamma_{0j}\xi_j^2 + 6\gamma_{1j}\xi_j - 6\gamma_{2j}) = 0. \tag{3.5}$$

The hoop stress N_2 and the axial moment M_1 calculated by (2.3) with (3.4) taken into account are

$$N_2 = \frac{Eh\alpha}{4}\sum_{j=1}^{n}[-\gamma_j(\cos(\xi - \xi_j) + \sin|\xi - \xi_j|) + 2\gamma_{0j}\sin(\xi - \xi_j) +$$
$$+ 2\gamma_{1j}(\cos(\xi - \xi_j) - \sin|\xi - \xi_j|) - 4\gamma_{2j}\operatorname{sgn}(\xi - \xi_j)\cos(\xi - \xi_j)]\,e^{-|\xi - \xi_j|}, \tag{3.6}$$

$$M_1 = -\frac{Eh\alpha R}{8a^2}\sum_{j=1}^{n}\{2\gamma_j|\xi-\xi_j| + \gamma_j(\cos(\xi-\xi_j) - \sin|\xi-\xi_j|)e^{-|\xi-\xi_j|} +$$
$$+ 2\gamma_{0j}(1 - e^{-|\xi-\xi_j|}\cos(\xi-\xi_j))\operatorname{sgn}(\xi-\xi_j) + [2\gamma_{1j}(\cos(\xi-\xi_j) +$$
$$+ \sin|\xi-\xi_j|) - 4\gamma_{2j}\sin(\xi-\xi_j)]e^{-|\xi-\xi_j|}\}. \tag{3.7}$$

It will be seen from (3.3) and (3.6) that the obtained extremum distribution of temperature T and stresses N_2 is defined by piecewise-continuous functions. A distribution of T and N_2 is continuous with respect to ξ, when $\gamma_{2j}=0$ is assumed.

Let us consider the particular case of the solution applicable to the simplest conditions of localized heating. Let the locally heated zone be bounded by sections $\xi=\pm\eta$, at which the temperature is zero. Temperature T attains its maximum equal T_0 at section $\xi=0$.

In this problem the extremum temperature field (3.3), which satisfies the condition of symmetry with respect to section $\xi=0$ and is continuous as well as its first derivative, is defined by

$$T = T_0[2|\xi/\eta|^3 - 3(\xi/\eta)^2 + 1] \quad (|\xi| \leqslant \eta), \quad T = 0 \quad (|\xi| \geqslant \eta). \tag{3.8}$$

The ring stress and the axial moment corresponding to (3.8) are defined by

$$N_2 = \frac{3Eh\alpha T_0}{\eta^3}[(\cos(\xi+\eta) + \sin|\xi+\eta|)e^{-|\xi+\eta|} + (\cos(\xi-\eta) + \sin|\xi-\eta|)e^{-|\xi-\eta|} -$$
$$- 2(\cos\xi + \sin|\xi|)^{-|\xi|} + \eta(e^{-|\xi+\eta|}\sin(\xi+\eta) - e^{-|\xi-\eta|}\sin(\xi-\eta))], \tag{3.9}$$

$$M_1 = \frac{3Eh\alpha T_0}{2a^2\eta^3}[2|\xi+\eta| + 2|\xi-\eta| - 4|\xi| + (\cos(\xi+\eta) - \sin|\xi+\eta|)e^{-|\xi+\eta|} +$$
$$+ (\cos(\xi-\eta) - \sin|\xi-\eta|)e^{-|\xi-\eta|} - 2(\cos\xi - \sin|\xi|)e^{-|\xi|} -$$
$$- \eta(1 - \cos(\xi+\eta)e^{-|\xi+\eta|})\operatorname{sgn}(\xi+\eta) + \eta(1 - \operatorname{Cos}(\xi-\eta)e^{-|\xi-\eta|})\operatorname{sgn}(\xi-\eta)]. \tag{3.10}$$

Curves

$$N^* = \frac{N_2}{Eh\alpha T_0}, \qquad M^* = \frac{a^2 M_1}{Eh\alpha R T_0}$$

are shown in Fig. 1 for $\nu=0.3$ and R/h=20 and 40 with the width of the heated zone equal to the cylinder radius ($\eta=a/2$). Curves of these magnitudes for a heated-zone width equal to the cylinder diameter ($\eta=\alpha$) are given in Fig. 2. The profiles of the temperature fields $T^*=T/T_0$ are shown in these by dashed lines.

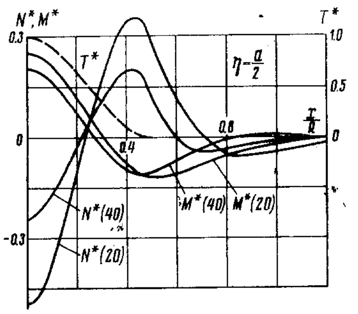

Fig. 1

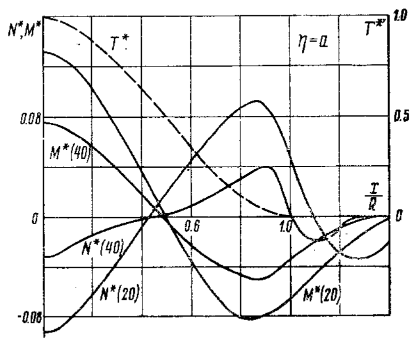

Fig. 2

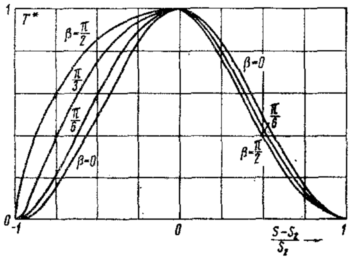

Fig. 3

4. In a conical shell the extremum temperature fields satisfy Eq. (2.9), whose general solution is of the form

$$T = \frac{\mathrm{ctg}\,\beta}{2\alpha m \sin\beta} \sum_{j=1}^{n} \left[\lambda_j \left(s \ln \frac{s_j}{s} + \frac{3}{2} s + \frac{s_j^2}{2s} - 2s_j\right) + \lambda_{0j}\left(\frac{s}{s_j} + \frac{s_j}{s} - 2\right) + \lambda_{1j}\left(\frac{s}{s_j^2} - \frac{1}{s}\right) + 2\lambda_{2j}\frac{s}{s_j^3}\right] S_+(s_j - s) + C_0 + C_1 s + \frac{C_2}{s}, \tag{4.1}$$

where C_0, C_1, and C_2 are arbitrary constants. The extremum solution obtained is a piecewise-continuous function. A continuous temperature distribution obtains for $\lambda_{2j}=0$. Should the continuity of the first derivative also be required, $\lambda_{ij}=0$ must be stipulated.

Let us consider an infinite conical shell closed at its vertex ($s_0=0$, $s_*=\infty$). We assume the temperature at the vertex and at infinity to be zero. It is then necessary to use solution (4.1), $C_0=0$, $C_1=0$, and $C_2=0$, and impose on coefficients λ_{ij} and λ_j the following conditions

$$\sum_{j=1}^{n} \lambda_j = 0, \qquad \sum_{j=1}^{n} \left(\lambda_j \ln \frac{s_j}{s_k} + \lambda_{0j}\frac{1}{s_j} + \frac{\lambda_{1j}}{s_j^2} + \frac{2\lambda_{2j}}{s_j^3}\right) = 0,$$
$$\sum_{j=1}^{n} (\lambda_j s_j + \lambda_{0j}) = 0, \qquad \sum_{j=1}^{n} \left(\frac{\lambda_j}{2} s_j^2 + \lambda_{0j} s_i - \lambda_{1j}\right) = 0 \qquad (0 < s_k < \infty). \tag{4.2}$$

We separate from (4.1) the twice differentiable extremum temperature field which locally heats zone $s_1 \le s \le s_3$ with the following conditions:

$$T(s_1) = 0, \quad T(s_2) = T_0, \quad T'(s_2) = 0, \quad T(s_3) = 0, \tag{4.3}$$

where $0 < s_2 \le s_3$. This solution is of the form

$$T = T_0 \left\{\left[a_2\left(\frac{s}{s_2}\ln\frac{s_2}{s} + \frac{3}{2}\frac{s}{s_2} - 2 + \frac{s_2}{2s}\right) + a_{02}\frac{(s_2 - s)^2}{s_2 s}\right] S_+(s_2 - s) + \left[a_3\left(\frac{s}{s_3}\ln\frac{s_3}{s} + \frac{3}{2}\frac{s}{s_3} - 2 + \frac{s_3}{2s}\right) + a_{03}\frac{(s_3 - s)^2}{s_3 s}\right] S_+(s_3 - s)\right\}. \tag{4.4}$$

Here

$$a_2 = -\frac{(s_1 + s_2)s_2}{s_1^2 - s_2^2 + 2s_1 s_2 \ln(s_2/s_1)} - \frac{(s_2 + s_3)s_2}{s_2^2 - s_3^2 + 2s_3 s_2 \ln(s_3/s_2)}\Bigg),$$

$$a_{02} = -\frac{1}{2}\left(\frac{s_1}{s_2 - s_1} + \frac{s_3}{s_3 - s_2} - \frac{s_2^2 - s_2^1}{s_1^2 - s_2^2 + 2s_1 s_2 \ln(s_2/s_1)} + \frac{s_3^2 - s_2^2}{s_2^2 - s_3^2 + 2s_2 s_3 \ln(s_3/s_2)}\right.,$$

(4.5)

$$a_3 = \frac{(s_2 + s_3)s_3}{s_2^2 - s_3^2 + 2s_2 s_3 \ln(s_3/s_2)}, \qquad a_{03} = \frac{s_3}{2(s_3 - s_2)} \frac{2s_2^2 \ln s_3/s_2 - s_3^2 + s_2^2}{s_2^2 - s_3^2 + 2s_2 s_3 \ln(s_3/s_2)}. \tag{4.5}$$

The numerical investigation of (4.4) was carried out for a heated-zone width equal to the diameter of cross section $s=s_2$ ($s_3-s_1=2R$, where R is the cross-section radius) and $s_3-s_2=s_2-s_1$. Curves of temperature $T^*=T/T_0$ are shown in Fig. 3 in terms of coordinate $s^*=(s-s_2)/s_2$ for several values of β. The case of $\beta=0$ corresponds to a cylindrical shell of radius R heated by the temperature field (3.8). The other limit case ($\beta=1/2\pi$) corresponds to that of the plane development of a conical shell.

The analysis carried out above of extremum temperature fields in the examples of cylindrical and conical shells was confined to the simplest conditions for the localized heating, under which limits are imposed on the variation of temperature at specified sections of the shell. Extremum solutions for more general conditions, which include supplementary conditions imposed on the level of temperature-induced stresses, can be similarly derived. Such conditions can be satisfied by a suitable choice of parameters

λ_{ij}, λ_j, and s_j appearing in the Euler equation (1.12). We point out that the temperature-field determination would, in that case, necessitate taking into consideration the complete set of equations of the problem.

LITERATURE CITED

1. É. I. Grigolyuk, Ya. I. Burak, and Ya. S. Podstrigach, "An extremum problem of thermal resilience for an infinite cylindrical shell," Dokl. AN SSSR, vol. 174, no. 3, 1967.
2. Ya. S. Podstrigach and S. Ya. Yarema, Temperature-Induced Stresses in Shells [in Ukrainian], Izd-vo AN UkrSSR, 1961.
3. I. M. Gel'fand and S. V. Fomin, Calculus of Variations [in Russian], Fizmatgiz, Moscow, 1961.
4. I. M. Gel'fand and G. E. Shilov, Generalized Functions and Operations upon them [in Russian], Fizmatgiz, Moscow, 1959.